STAT 2593

Lecture 008 - Conditional Probability

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Conditional Probability

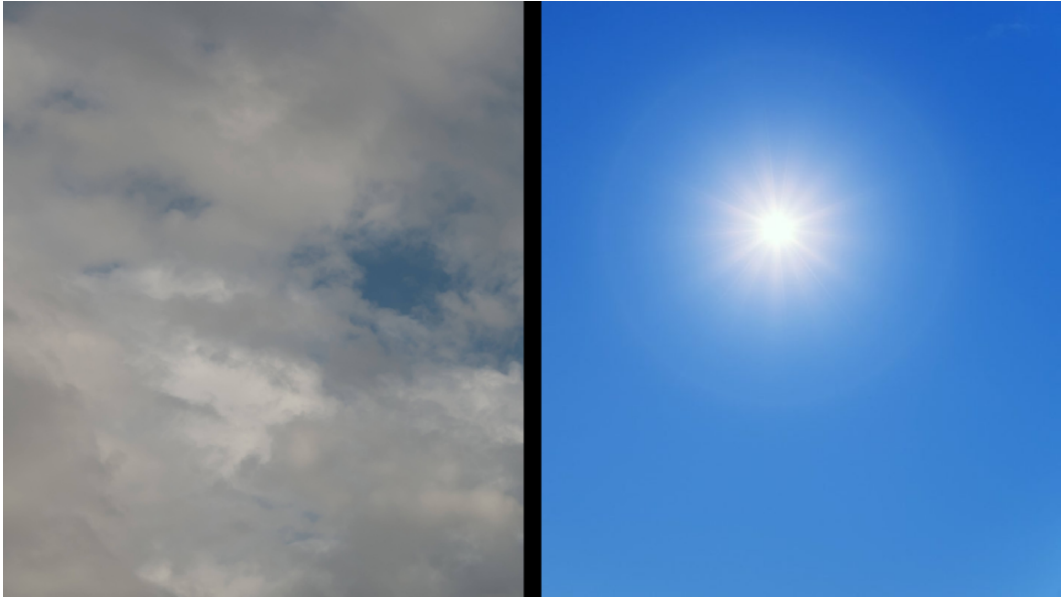
Learning Objectives

1. Understand what conditional probability is, and how to compute it.

2. Understand the multiplication rule for probability.

3. Understand the law of total probability.

4. Understand Bayes' Theorem and its use cases.



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    - Now $N = 4$.

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- This can be re-arranged for the **multiplication rule** which states

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- This is called the **law of total probability**.
- Often easier to count conditional probabilities rather than **marginal probabilities** directly.

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- For instance...
  - How likely is it to have a rare disease if you test positive for it?
  - How likely is an email to be spam, given that it was detected by the filter?

Summary

- Conditional probabilities consider information that we already know.
- Conditional probabilities give rise to the multiplication rule, and the law of total probability.
- These tools help to compute marginal probabilities.
- Bayes' Theorem allows us to "update our view" on the world, by combining conditional probability with the multiplication rule.